Total number of printed pages – 7

2023

MATHEMATICS

Full Marks : 100

Pass Marks : 33

Time : Three hours

Attempt all Questions.

The figures in the right margin indicate full marks for the questions.

For Question Nos. 1 – 4, write the letter associated with the correct answer.

1. The value of $\cot^{-1}\left(\cot\frac{7\pi}{6}\right)$ is : 1

A. $\frac{7\pi}{6}$ B. $\frac{5\pi}{6}$

C. $\frac{\pi}{3}$ D. $\frac{\pi}{6}$

2. If $\frac{dy}{dx} = \frac{y}{x}$, then $\frac{d^2y}{dx^2}$ equals : 1

A. 0

B. 1

C. –1

D. –2

3. The degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 + 2x^2\left(\frac{d^2y}{dx^2}\right)^3 + 3x\left(\frac{dy}{dx},\right)^4 + y = 0$ is : 1

A. 1

B. 2

C. 3

D. 4

4. The distance of the plane $3x + 4y + 12z = 26$ from the origin is : 1

A. 26

B. 13

C. 2

D. 1

5. Find the identity element of the binary operation * on R defined by $a*b = \frac{ab}{5} \forall a, b \in R$. 1

6. Write down the range of the function $\tan^{-1}$. 1

7. Find the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$. 1

8. If $A=\begin{bmatrix}-1 & -1\\ k & 2\end{bmatrix}$ and $A^2=A$, find the value of k. 1

9. Find the value of k for which the function 1

$$f(x)=\begin{cases}\dfrac{\sin 3x}{4x}, & x\neq 0\\ k, & x=0\end{cases}$$

is continuous at $x=0$.

10. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$. 1

11. If $f(x)=\int_0^x t\sin t\,dt$, write down the value of $f'(x)$. 1

12. What is meant by the general solution of a differential equation ? 1

13. When is a function $f(x)$ said to be differentiable at $x=a$? 1

14. State chain rule for finding the derivative of a composite function. 1

15. Prove that $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]=\tan^{-1}\left(\frac{a}{b}\right)-x, \left(\frac{a}{b}\tan x>-1\right)$ 2

16. If $y=5\cos x+3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$. 2

17. The length x of a rectangle is decreasing at the rate of 5cm/minute and the width y is increasing at the rate of 4cm/minute. Find the rate of change of the area of the rectangle when $x=8$cm and $y=6$cm. 2

18. Evaluate : $\int \frac{\cos x-\sin x}{1+\sin 2x}\,dx$. 2

19. What is meant by a homogenous differential equation ? Describe how it can be reduced to a form in which the variables are seperable. 2

20. Evaluate : $\int_0^2 x\sqrt{2-x}\,dx$. 2

21. Show that the vectors $\vec{a}=\hat{i}+3\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}-\hat{k}$ and $\vec{c}=7\hat{j}+3\hat{k}$ are coplanar. 2

22. Find the angle between the pair of lines given by

$\vec{r}=(3\hat{i}+2\hat{j}-4\hat{k})+\lambda(\hat{i}+2\hat{j}+2\hat{k})$ and $\vec{r}=(5\hat{i}-2\hat{j})+\mu(3\hat{i}+2\hat{j}+6\hat{k})$. 2

23. Find the equation of the plane through the intersection of the planes

$3x+y+2z+4=0$ and $x+y+z+2=0$ and the point $(2,-2,1)$. 2

24. Let A = N × N and * be the binary operation on A defined by $(a,b)*(c,d)=(a+c,b+d)\ \forall(a,b),(c,d)\in A$. 4

25. If $x=a(\cos t+t\sin t)$ and $y=e(\sin t-t\cos t)$, find $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{3}$: $\frac{d^2y}{dx^2}$.

Or

Find $\frac{dy}{dx}$, if $x^y+y^x=a^b$.

4

26. Prove that $\int\sqrt{x^2-a^2}\,dx=\frac{1}{2}\left[x\sqrt{x^2-a^2}-a^2\log|x+\sqrt{x^2-a^2}|\right]+C$ 4

27. Evaluate : $\int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx$

Or

$\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$ 4

28. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. 4

29. Find the particular solution of the differential equation $\frac{dy}{dx}+y\cot x = 4x \operatorname{cosec} x (x \neq 0)$, given that $y=0$ when $x=\frac{\pi}{2}$. 4

30. Derive the vector equation of a line passing through two points. 4

31. If A and B are two independent events such that P(A)= $\frac{1}{4}$ and P(B) = $\frac{1}{2}$, find (i) P (not A and not B) (ii) P (A'/B'). 4

32. Define symmetric and skew symmetric matrices. For any square matrix A with real number entries, prove that $A+A'$ is a symmetric matrix and $A-A'$ is a skew symmetric matrix and hence deduce that any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix. 6

33. Solve the following system of equations by matrix method:

$2x+3y+3z=5$

$x-2y+z=-4$

$3x-y-2z=3$. 6

34. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base. 6

Or

Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

35. Find the area of the triangle whose vertices are (1,1,2), (2,3,5) and (1,5,5). 6

Or

If two medians of a triangle are equal, prove by vector method that the triangle is isosceles.

36. One kind of cake requires 200 g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes. 6

Or

A diet is to contain at least 80 units of vitamin A 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs. 20 per unit food and F_2 costs Rs. 30 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals.

Contd.

One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

37. An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. Find which is more likely to happen that the second ball is red and that the second ball is black. 6