2019
MATHEMATICS

Full Marks : 100

Pass Marks : 33

Time : Three hours

Attempt all Questions.

The figures in the right margin indicate full marks for the questions.

*For Question Nos. **1-6**, write the letter associated with the correct answer.*

1. Let R_1 be a relation defined over the set of real numbers as $R_1 = \{(a,b): a \geq b, a,b \in R\}$. Then R_1 is 1

   A. an equivalence relation.

   B. reflexive but not symmetric.

   C. reflexive, symmetric but not transitive.

   D. symmetric but not reflexive, transitive.

2. If $\alpha = \tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ and $\beta = \tan^{-1}\left(-\tan\frac{2\pi}{3}\right)$, then 1

   A. $4\alpha = 3\beta$

   B. $3\alpha = 4\beta$

   C. $\alpha - \beta = \frac{7\pi}{12}$

   D. $\alpha = \beta$.

3. The value of $\int \tan x \sec^3 x \, dx$ is 1

A. $\sec x \tan x + C_1$

B. $3 \sec^2 x + C_1$

C. $\frac{1}{3}\sec^3 x + C_1$

D. $\frac{1}{3}\tan^2 x + C_1$.

4. The general solution of the differential equation $\frac{ydx - sdy}{y} = 0$ is 1

A. $xy = C_1$

B. $x = C_1 y^2$

C. $y = C_1 x$

D. $y = C_1 x^2$.

5. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors then the projection vector of $\vec{a}$ on $\vec{b}$ is 1

A. $\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}\cdot\vec{a}$

B. $\frac{(\vec{a}\cdot\vec{b})}{|\vec{b}|^2}\cdot\vec{b}$

C. $\frac{(\vec{a}\times\vec{b})}{|\vec{a}|^2}\cdot\vec{a}$

D. $\frac{(\vec{a}\times\vec{b})}{|\vec{b}|^2}\cdot\vec{b}$

6. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector then angle between $\vec{a}$ and $\vec{b}$ is 1

A. $\frac{\pi}{6}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{2}$

D. $\frac{\pi}{4}$.

7. Define a reflexive relation in a set. 1

8. If $4\sin^{-1} x + \cos^{-1} x = \pi$, then what is the value of x ? 1

9. Define a skew-symmetric matrix. 1

10. Find the surface area of a sphere when its volume is changing at the same rate as its radius. 1

11. Define monotonic function. 1

12. State triangle law of vectors. 1

13. Find $\vec{a} \cdot \vec{b}$ if $|\vec{a}| = 8$, $|\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 12$. 1

14. Define an integral or primitive of a given function. 1

15. Write the conditions that the line $\vec{r} = \vec{a} + \lambda \vec{b}$ lies in the plane $\vec{r} \cdot \vec{n} = d$. 1

16. In a binomial distribution, if the probability of getting success is $\frac{1}{4}$ and standard deviation is 3 then write its mean. 1

17. Show that the function $f : R \to R$ given by $f(x) = ax + b$, where $a, b \in R$, $a \neq 0$ is bijective. 3

18. Prove that $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$. 3

19. If $A=[a_{ij}]$ is a non-singular matrix then prove that $A^{-1}=\frac{1}{|A|}(adj\ A)$. 3

20. Show that 3

$$\begin{vmatrix} 3a & -a+b & -a+c \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca).$$

21. Define a differential equation. What do you mean by the order and degree of a differential equation ? 3

22. In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in Class-XII. What is the probability that a student chosen randomly studies in Class-XII given that the chosen student is a girl ? 3

23. If a function is differentiable at a point, prove that it is continuous at that point. 4

24. Using integration, find the area of the smaller region bounded by the curve $\frac{x^2}{16}+\frac{y^2}{9}=1$ and the line $\frac{x}{4}+\frac{y}{3}=1$. 4

25. Solve the differential equation

$$x\frac{dy}{dx}-y=\sqrt{x^2+y^2}$$ 4

OR

Solve the differential equation

$$\left(1-x^2\right)\frac{dy}{dx}-xy=x^2,\ \text{given that } y=2 \text{ when } x=0.$$

26. Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}\cdot\vec{d}=15$. 4

27. Prove that every first degree equation in x, y, z represents a plane. 4

28. Find the cartesian as well as vector equations of the planes through the intersection of the planes $\vec{r}\cdot\left(2\hat{i}+6\hat{j}\right)+12=0$ and $\vec{r}\cdot\left(3\hat{i}-\hat{j}+4\hat{k}\right)=0$ which are at a unit distance from the origin. 4

OR

Find the equation of the line passing through the point (2, 1, 3) and perpendicular to the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$.

29. Using elementary transformations, find the inverse of the matrix 6

$$A=\begin{bmatrix}1 & 3 & -2\\ -3 & 0 & -5\\ 2 & 5 & 0\end{bmatrix}$$

OR

Show that $A=\begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ satisfies the equation $A^3 - A^2 - 3A - I_3 = 0$.

Hence find A^{-1}.

30. If $f(x) = \begin{cases} ax^2 - b, & \text{if } |x| < 1 \\ \dfrac{1}{|x|}, & \text{if } |x| \geq 1 \end{cases}$

is differentiable at $x = 1$, find a, b. 6

OR

Find $\dfrac{dy}{dx}$, if $y = \sin(x^x) + (\sin x)^x + \sqrt{\sin x}$.

31. Find the volume of the greatest cylinder which can be inscribed in a cone of height 'h' and semi-vertical angle α. 6

32. Prove that: 6

(i) $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(ii) $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log\left|x + \sqrt{\sqrt{x^2 - a^2}}\right| + C.$

33. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\,dx}{1 + \sin x \cos x}$. 6

34. A company uses 3 machines to manufacture and sells two types of teaching aids A and B of mathematics. Machines M_1, M_2 and M_3 take 1 hour, 2 hours and $\frac{8}{5}$ hours to make a teaching aid A and 2 hours, 1 hours and $\frac{8}{5}$ hours to make a teaching aid B. The profit on each teaching aid A is Rs. 100 and on a teaching aid B is Rs. 150. No machine can work more than 40 hours per week. How many teaching aid of each type should be made to maximise the company's profit? Solve the L.P.P. graphically (Graph paper will not be supplied). 6

35. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}$, $\frac{1}{5}$, $\frac{1}{10}$ and $\frac{2}{5}$. The probability that he will be late are $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by bus ? 6

OR

In a game a man wins a rupee for a six and looses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/looses.